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Mathematical modeling of a two-echelon sustainable dynamic supply chain

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This technical note presents a perspective on the creation of value in time-delayed agri-food supply chains (AFSCs) and examines its impact on dynamic pricing, human resources, and sustainability. In general, analyzing AFSCs may seem straightforward; however, the complexity involved in managing dynamic pricing, sustainability trade-offs, and establishing ethnographic measures to improve worker productivity makes these challenges far from simple from an operational perspective. This study presents a mathematical model incorporating time delays in a two-tier supply chain consisting of producers and consumers, based on certain assumptions. In addition, a finite-time optimal control (OC) problem is addressed analytically using Pontryagin's maximum principle (PMP). Finally, stability and sensitivity analyses are conducted for a system that experiences losses in the production and consumption stages of the dynamic supply chain (DSC).

KEYWORDS
creation of value, dynamic supply chains, mathematical modeling, optimal control, time delays

1 Introduction

Supply chain engineering (SCE) applies quantitative, analytical, and mathematical methods based on engineering principles to optimize supply chains, thereby improving efficiency, resilience, and value creation across production, distribution, logistics, and final retailing (Simchi-Levi et al., 2008). In SCE, mathematical models incorporating time delays provide valuable insights for inventory management in dynamic supply chains (DSCs). Mathematical models with time delays play a critical role in understanding and optimizing DSCs in the agri-food sector, where processes such as production, transportation, processing, and market demand are interdependent and occur over extended periods. To synchronize supply and demand, companies must create value in the finished goods delivered to customers from suppliers (Martínez-Olvera and Davizon-Castillo, 2015). Agri-food supply chains (AFSCs) are experiencing profound structural and functional transformations over the past few decades, driven by globalization, technological innovation, evolving consumer preferences, and increasing concerns regarding sustainability and equity (Beg et al., 2017). These dynamics have reshaped production, processing, and distribution systems, giving rise to more complex and interconnected global networks. Within this context, cocoa (*Theobroma cacao Lin*) has emerged as a crop of notable cultural, nutritional, and strategic importance, particularly in tropical regions such as Mexico.

In economic and consumption terms, cocoa ranks among the world's most traded and consumed agricultural products, similar to coffee in market dynamics. Demand for cocoa-based products continues to grow, driven by population growth, urbanization, and diversification of consumer tastes, particularly in emerging economies (Mueller et al., 2023). However, the Mexican cocoa industry has undergone a process of gradual transformation. The states of Chiapas, Tabasco, Veracruz, Oaxaca, and Guerrero have consolidated as the country's principal production regions, each characterized by distinct agroecological conditions and production dynamics. Cocoa production is concentrated in the states of Tabasco and Chiapas, which together account for 61,344 hectares of cultivated land (Sánchez et al., 2019). In addition, cocoa is globally recognized not only as a key raw material for the confectionery and chocolate industries but also as a source of livelihoods for millions of smallholder farmers in Latin America, Africa, and Asia (Naranjo-Merino et al., 2018; Kouassi et al., 2021; Haynes et al., 2012).

The primary contributions and significance of this research are as follows:

1. The impact of cocoa supply chain (CSC) production and consumption in Mexico is analyzed, assuming low volume.
2. The creation of value in the cocoa supply chain is examined through price dynamics, ethnographic analysis, and sustainability impacts.
3. A mathematical model incorporating time delays is presented for the supply chain of producers and consumers, under certain assumptions.
4. An optimal control (OC) problem is addressed using Pontryagin's maximum principle (PMP) with a proper functional for minimization, which is solved analytically.
5. A stability analysis of the system is conducted, assuming inventory losses at each stage of the DSC.

The following research questions are addressed in this study:

1. Is the cocoa supply chain a suitable subject of analysis for value creation using proper sustainability metrics?
2. Does the dynamic pricing of the CSC constitute a relevant area of analysis within a supply chain analytics context?
3. Is time delay a valuable lead time factor to consider in low-volume, high-variability systems?
4. Is finite-time optimal control with proper initial and final conditions formulated via PMP an optimal approach for this CSC analysis?
5. Is stability analysis of the CSC a necessary condition for the proper evolution of the system?

The remaining article is organized as follows: Section 2 analyzes value creation in the CSC. Section 3 presents the methodology, emphasizing the transition from short food supply chains (SFSCs) to global markets, the role of dynamic pricing, sustainability trade-offs, and an ethnographic analysis to capture labor force impacts within the cocoa supply chain. Section 4 examines a case study focused on mathematical modeling, optimal control, and stability analysis. Section 5 presents a discussion, and Section 6 outlines future research directions for this research letter.

2 Methods

Agri-food sustainable DSCs refer to interconnected systems that effectively manage the production, processing, distribution, and consumption of agricultural and food products in ways that are environmentally sustainable and economically profitable. This framework highlights the importance of production, distribution, and marketing for the future development of sustainable supply chains. In Mexico, the CSC focuses on both increasing domestic production and reducing reliance on imports.

Value creation in the agri-food sector is crucial not only for the profitability of the industry but also for ensuring its sustainability. By focusing on economic, environmental, and social dimensions, we can build a food system that satisfies the needs of a growing global population while protecting the planet and promoting social equity. Value creation in this context goes beyond the generation of wealth; it encompasses improving livelihoods, fostering innovation, conserving natural resources, and addressing systemic inequalities (Table 1). Central to this vision are strategic planning and the integration of sustainable practices within the AFSC (Figure 1).

2.1 From short food supply chains to global markets: raw material flow

By definition, a short food supply chain (SFSC) is a system that emphasizes local production, transparency, and sustainability while minimizing the number of intermediaries between producers and consumers along the supply chain. Based on this, it is proposed to strengthen SFSCs by establishing cooperatives where producers focus on adding value to their products, transitioning cocoa through its different stages to become producers and manufacturers, and ultimately reaching end customers with branded finished products. However, several persistent issues in the field continue to undermine these efforts.

Consequently, the sustainability and competitiveness of the CSC have become central issues in agricultural sustainable research, encompassing environmental management, labor conditions, and fair trade practices. In general, a commercial chocolate bar contains only 25%—35% cocoa mass; cocoa in semisweet chocolate bars range from 45 to 99%, while white chocolate bars only contain cocoa butter, milk, and sugar (Duana-Ávila et al., 2023). In addition, the low application of fertilizers and improper management of shade conditions within plantations limit productivity and quality. Currently, researchers are editing the cocoa genome to produce disease-resistant plants (Fister et al., 2018). As global demand for high-quality cocoa continues to rise, Mexico's cocoa-producing regions will remain critical players in the international market, especially if they continue to prioritize sustainability and quality.

2.2 Mathematical modeling of dynamic supply chains with time delays

The decision-making problem of the producer and retailer under time delay effects in a two-echelon supply chain was

TABLE 1 Comparison of the two-tier cocoa supply chain from producer and consumer perspectives.

Characteristic	Producer perspective	Consumer perspective
Creation of value	Based on production analysis and quality	Brand definition and consumer feedback
Objective	Maximize cocoa production and improve profitability	High-quality chocolate products
Sustainability trade-offs	Reduction of waste and improvement of farm productivity	Environmental aspects positive and negative externalities
Market spot	Dynamic pricing determined by global markets	Chocolate markets reflect quality and social impacts
Mandatory structure	Supply-driven (production, logistics, and exportation)	Demand-driven (Ethical consumption)

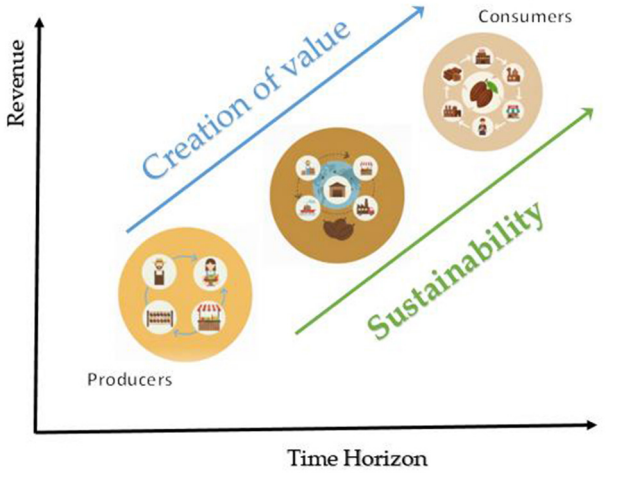

FIGURE 1
Value creation in the cocoa supply chain.

examined by Li et al. (2024). Time delays are intrinsic to many stages of the AFSC and can significantly affect decision-making, efficiency, and overall system performance. In the context of modern supply chains, a dynamic system is one that is continually shaped by various fluctuating factors, such as production rates, consumer demand shifts, transportation delays, and external influences including market conditions, economic trends, and unforeseen weather events. These variables interact in real time. Consequently, businesses must remain agile and responsive, adjusting their strategies and operations as needed to sustain efficiency and mitigate risks.

In the context of cocoa production, for example, delays can arise in areas such as crop growth, harvesting schedules, processing, and market feedback. Yu et al. (2020) addressed optimal control for multiple delays in supply chains. By accounting for time delays, stakeholders can make more informed decisions, improve efficiency, reduce costs, and enhance the resilience of the supply chain. As the global agri-food sector faces growing challenges, such as climate change, market volatility, and shifting consumer preferences, the role of mathematical modeling in managing time delays has become more critical for achieving sustainable and efficient supply chains.

2.3 Dynamic pricing: the cash flow

Dynamic pricing models have gained relevance as tools to address volatility and improve value for cocoa farmers. These models, however, face several key limitations: (1) market structure constraints; (2) institutional and socioeconomic issues; (3) environmental concerns; (4) mathematical modeling limitations, such as biological delays in cocoa production (trees take 3–5 years to mature); and (5) positive and negative externalities in production. Therefore, mathematical modeling has emerged as a critical approach for analyzing and optimizing these systems, enabling better decision-making under uncertainty and enhancing supply chain resilience. Techniques such as ordinary differential equations (ODEs) (Davizón et al., 2014), system dynamics, and optimal control theory have been applied to model inventory levels, production rates, and pricing strategies in DSCs (Mehrbanfar et al., 2020). Therefore, dynamic pricing in the CSC faces both practical and structural limitations due to the characteristics of agricultural commodity markets and the socioeconomic context of low-volume cocoa production.

2.4 Labor force: ethnographic analysis

Supply chain ethnography refers to a qualitative analysis, from a sociological perspective, of the practices of the social and cultural environment where the processes of planting, harvesting, and production are carried out in agricultural fields by workers, farmers, and traders. Therefore, it is important to understand the ethnography of cocoa producers in Mexico from a qualitative perspective for supporting industrialization efforts aimed at adding and creating value within the CSC. Furthermore, it is essential to highlight the ethnographic Key Performance Indicators (KPIs) as defined by Rasmussen et al. (2020): the behavioral dimension (socio-cultural factors), the structural dimension (interactions among supply chain participants), and the contextual dimension (market and environmental variables). The working conditions of cocoa producers can be improved through ethnographic analysis, which examines the social, cultural, and economic factors affecting workers. Through the application of surveys and interviews, this approach can identify labor inequalities, leverage practices, and informal contractual trade-offs along the CSC.

2.5 Sustainability trade-offs

Sustainability is the ability to meet present needs without limiting future generations, which is achieved by balancing environmental, social, and economic priorities (Giddings et al., 2010). In addition, an AFSC encompasses the "farm-to-fork" activities from production to marketing. In this context, sustainability involves managing these stages to reduce environmental impact, promote social equity, and ensure economic viability (De Silva et al., 2023). Therefore, the CSC is a complex system where sustainability addresses trade-offs, which requires coordination among farmers, enterprises, and government, adopting a systemic perspective. This perspective incorporates life cycle assessment [a method to validate the environmental impact of a product or service across its entire life cycle (from raw material to end customer)], socioecological analysis (the study of interactions between human societies and the environment, aimed at understanding and managing the sustainability of social and ecological systems), and ethnographic studies (analysis of local cocoa supply chain behaviors, as previously discussed) (Yuan et al., 2024).

3 Results

Inventory management in DSCs is critical for ensuring smooth operations, minimizing costs, and meeting demand effectively (Taboada et al., 2022). A dynamic supply chain refers to a system that is constantly influenced by variables such as production rates, demand fluctuations, transportation delays, and external factors, including market conditions and weather events. Managing inventory in such an environment requires flexibility and real-time data. Here, we discuss the key aspects of inventory management in dynamic supply chains (Benítez-García et al., 2025).

3.1 Mathematical modeling

A two-echelon dynamic supply chain was analyzed by Tshinangi et al. (2025), with similar approaches applied to two-echelon cold supply chains (Acerce and Denizhan, 2025; Sebatjane, 2025), using the following ordinary differential equation, which considers both producers and consumers in a DSC:

$$\frac{dI}{dt} = S\left(t-\theta\right) - d\left(t\right) \tag{1}$$

Equation 1 represents a differential equation that relates the inventory level (I), production rate S(t), and demand rate d(t). A proper lead time (θ) is considered because CSCs are related to time delay approaches for perishable inventory systems.

In general, the cocoa dynamic supply chain structure can be expressed as:

$$\frac{dI_1}{dt} = S_1\left(t-\theta\right) - d_1\left(t\right) \tag{2}$$

$$\frac{dI_2}{dt} = S_2\left(t-\tau\right) - d_2\left(t\right) \tag{3}$$

To present a model reduction for the system, in general, we have the following equation:

$$\frac{dI_1}{dt} = S_1\left(t-\theta\right) \tag{4}$$

$$\frac{dI_2}{dt} = -d_2\left(t\right) \tag{5}$$

Applying the Taylor series to the time delay term, we have (Insperger, 2015):

$$S_1\left(t-\theta\right) \approx S_1\left(t\right) - \theta\dot{S_1} \tag{6}$$

By applying Equations 4–6, we have:

$$\frac{dI_1}{dt} = S_1\left(t\right) - \theta\frac{dS_1}{dt} \tag{7}$$

$$\frac{dI_2}{dt} = -d_2\left(t\right) \tag{8}$$

Assuming an equilibrium condition where $S_1\left(t\right) = d_2\left(t\right)$. The final system is as follows:

$$\frac{dI_1}{dt} = S_1\left(t\right) - \theta\frac{dS_1}{dt} \tag{9}$$

$$\frac{dI_2}{dt} = -S_1\left(t\right) \tag{10}$$

By adding Equations 9, 10, we obtain:

$$\frac{dI_1}{dt} + \frac{dI_2}{dt} = -\theta\frac{dS_1}{dt} \tag{11}$$

In general, we have:

$$I_1\left(t\right) + I_2\left(t\right) = -\theta S_1\left(t\right) \tag{12}$$

Assuming a decreasing production rate of the form $S_1\left(t\right) = -kt$, and applying in Equation 12 for k > 0, we obtain:

$$I_1\left(t\right) + I_2\left(t\right) = k\theta t \tag{13}$$

Equation 13 applies the derivative of both sides of the equation:

$$\dot{I_1}\left(t\right) + \dot{I_2}\left(t\right) = k\theta \tag{14}$$

Equation 14 implies that the rate of change (the sum of variations in both inventories) is constant and equal to the net production with lead times (time delay). This reflects a steady-state scenario where the supply chain is in equilibrium. Under this condition, the production rate, adjusted for lead times, matches the demand and consumption rate, ensuring that inventory levels fluctuate predictably. After analyzing the two-tier cocoa supply chain, further applications for improving the efficiency, sustainability, and equity of the entire DSC are summarized in Table 2.

TABLE 2 Areas of contribution in the DSC.

Article	Description	Contribution area
Bhadoriya et al. (2021)	Carbon taxes are a key government strategy for addressing global warming. By penalizing emissions, these regulations have successfully lowered the carbon footprint of various industries	Sustainability engineering
Jani et al. (2023)	Payment policies tailored to a specific product type vary from general industry standards. For instance, perishable goods require different considerations because they expire much faster than standard, long-lasting products	Sustainability engineering
Chaudhari et al. (2023)	This study presents an inventory model tailored for perishable goods under a 'down-cash-credit' (DCC) payment framework. In this model, consumer demand is driven by price points, available inventory, and promotional efforts, all while accounting for the financial impact of a fixed carbon tax	Civil engineering
Spiegler et al. (2012)	This research moves beyond qualitative definitions, proposing a mathematical framework for assessing how supply chains respond to external shocks and disruptions	Supply chain engineering
Aramyan et al. (2007)	The study proposes a performance measurement system based on four key categories: efficiency, Flexibility, Responsiveness, and Food Quality	Supply chain engineering
Disney and Towill (2003)	This study applies control engineering techniques to mathematically analyze the bullwhip effect (the amplification of demand variability) as it moves upstream in a supply chain. It investigates the relationship between ordering policies, inventory variance, and system stability in a linear, continuous-time domain	Supply chain engineering

3.2 Optimal control for time-delayed DSCs

In previous studies, the application of OC in DSCs (Kappelman and Sinha, 2021) has been addressed. For the system dynamics described in Equations 9, 10, we have:

$$\frac{dI_1}{dt} = u(t) - \theta\frac{du}{dt} \tag{15}$$

$$\frac{dI_2}{dt} = -u(t) \tag{16}$$

Therefore, the optimal control (OC) problem for this system is:

$$\min_u J = \int_0^T \left(u^2 + \left(\frac{du}{dt}\right)^2\right)dt \tag{17}$$

s.t.

$$\frac{dI_1}{dt} = u(t) - \theta\frac{du}{dt}$$

$$\frac{dI_2}{dt} = -u(t)$$

With the following initial and final conditions in a time horizon of T = 100 days, MT refers to metric tons. Assuming an artisanal production rate in rural Mexico, the average inventory level is 200 kilograms (Kgs).

$$I_1(0) = 100\,MT \qquad I_1(100) = 0$$

$$I_2(0) = 200\,MT \qquad I_2(100) = 0$$

Applying PMP to solve the OC problem in Equation 17, we define the Hamiltonian as:

$$H = u^2 + \left(\frac{du}{dt}\right)^2 + \lambda_1\left(u - \theta\frac{du}{dt}\right) + \lambda_2(-u) \tag{18}$$

Here, λ_1 and λ_2 are Lagrange multipliers for the costates. Applying the Hamilton equations, we have:

$$\frac{dH}{du} = 0 \tag{19}$$

From Equation 19, we have:

$$u = \frac{-(\lambda_1 - \lambda_2)}{2} \tag{20}$$

In addition, the second condition follows:

$$\frac{dH}{d\dot{u}} = 0 \tag{21}$$

Therefore, we have:

$$2\dot{u} - \theta\lambda_1 = 0 \tag{22}$$

By integrating Equation 22 with respect to time, we have:

$$u(t) = \frac{\lambda_1\theta}{2}t + K \tag{23}$$

By substituting $u(t)$ into Equation 20, we have:

$$\lambda_2 = \lambda_1(1 + \theta t) + 2K \tag{24}$$

Considering that:

$$I_2(t) = -\int u(t)\,dt = -\int\left(\frac{\lambda_1\theta}{2}t + K\right)dt \tag{25}$$

Finally, we have:

$$I_2(t) = \frac{\lambda_1\theta}{2}t^4 - Kt + W \tag{26}$$

Applying the initial condition $I_2(0) = 200$, where W = 200.

In a similar manner, we can calculate $I_1(t)$ to achieve:

$$I_1(t) = \frac{\lambda_1\theta}{2}t^2 + K - \frac{\lambda_1\theta^2}{2} \tag{27}$$

After integrating Equation 27 with respect to time and applying the initial condition $I_1(0) = 100$, we obtain:

$$I_1(t) = \frac{\lambda_1\theta}{2}t^2 + \left(K - \frac{\lambda_1\theta^2}{2}\right)t + 100 \tag{28}$$

We can apply the final conditions. After applying algebra, we obtain:

$$2,500\lambda_1\theta + 100K - 50\lambda_1\theta^2 + 100 = 0 \tag{29}$$

$$-2,500\lambda_1\theta - 100K + 200 = 0 \tag{30}$$

Adding Equations 29, 30, we have:

$$\lambda_1 = \frac{6}{\theta^2}$$

By substituting $\lambda_1 = \frac{6}{\theta^2}$ into Equation 30, we have:

$$K = 2 - \frac{15,000}{\theta} \tag{31}$$

Finally, we have:

$$u(t) = \frac{3t - 150}{\theta} + 2 \tag{32}$$

$$I_1(t) = \frac{3}{2\theta}t^2 - \left(\frac{150}{\theta} + 1\right)t + 100 \tag{33}$$

$$I_2(t) = \frac{-3}{2\theta}t^2 + \left(\frac{150}{\theta} - 2\right)t + 200 \tag{34}$$

Thus, the optimal solution and the evolution of the inventory levels depend on time delay θ.

3.3 Stability analysis

The stability analysis of supply chains as dynamical systems relies on distinct mathematical approaches with proper system architecture. The Routh–Hurwitz stability criterion provides a necessary and sufficient condition for the stability of linear, continuous-time, single-input and single-output (SISO) systems. However, to address the inherent complexities of non-linear supply chain dynamics, Lyapunov's direct and indirect methods are employed, offering a generalized framework for evaluating equilibrium stability without explicitly solving the system's differential equations. Within this framework, to evaluate system stability under time delays and quantify inventory fluctuations, we propose a model that accounts for losses during both the production and consumption phases of the supply chain.

$$\frac{dI_1}{dt} = S_1(t) - \theta\frac{dS_1}{dt} - \alpha_1 I_1(t) \tag{35}$$

$$\frac{dI_2}{dt} = -S_1(t) - \alpha_2 I_2(t) \tag{36}$$

Theorem 1.

The system described in Equations 35, 36 is asymptotically stable if the conditions $\alpha_1 > 0$ and $\alpha_2 > 0$ are satisfied.

Proof.

We have a system of the form in the Laplace transform:

$$sI_1(s) = S_1(s) - \theta s S_1(s) - \alpha_1 I_1(s) \tag{37}$$

$$sI_2(s) = -S_1(t) - \alpha_2 I_2(s) \tag{38}$$

Presenting Equations 35, 36 in transfer function form, we have:

$$\frac{I_1(s)}{S_1(s)} = \frac{1 - \theta s}{s + \alpha_1} \tag{39}$$

$$\frac{I_2(s)}{S_1(s)} = \frac{-1}{s + \alpha_2} \tag{40}$$

For the transfer function Equation 39, the system is asymptotically stable if $\alpha_1 > 0$, which is based that the pole in the left is on the complex plane $s = -\alpha_1$. In addition, for the transfer function Equation 40, the system is asymptotically stable if $\alpha_2 > 0$, which is based that the pole in the left complex plane is on $s = -\alpha_2$.

3.4 Sensitivity analysis

To present a sensitivity analysis of the system dynamics of the DSC, we consider the following:

1) Partial derivatives with respect to θ:

$$\frac{\partial\dot{I}_1}{\partial\theta} = -\dot{u}, \frac{\partial\dot{I}_2}{\partial\theta} = 0, \text{and } \frac{\partial\left(\dot{I}_1 + I_2\right)}{\partial\theta} = k \tag{41}$$

2) Partial derivatives with respect to k:

$$\frac{\partial\dot{I}_1}{\partial k} = 0, \frac{\partial\dot{I}_2}{\partial k} = \theta_0, \frac{\partial\left(\dot{I}_1 + I_2\right)}{\partial k} = 0, \text{and } \frac{\partial\dot{u}}{\partial k} = -1 \tag{42}$$

Assuming the operating point:

$$u = u_o,\ \dot{u} = \dot{u}_o = -k,\ \theta = \theta_o\ , \text{and } k = k_o$$

The partial derivatives evaluated at the operating point: $u = 1,000$, $\dot{u}_o = -100$, $\theta = 20$, and $k = 100$

$$\frac{\partial\dot{I}_1}{\partial\theta} = 100, \frac{\partial\dot{I}_2}{\partial\theta} = 0, \frac{\partial\left(\dot{I}_1 + I_2\right)}{\partial\theta} = 100, \text{and } \frac{\partial\dot{u}}{\partial\theta} = 0 \tag{43}$$

Partial derivatives with respect to k:

$$\frac{\partial\dot{I}_1}{\partial k} = 0, \frac{\partial\dot{I}_2}{\partial k} = 0, \frac{\partial\left(\dot{I}_1 + I_2\right)}{\partial k} = 20, \text{and } \frac{\partial\dot{u}}{\partial k} = -1 \tag{44}$$

Partial derivatives with respect to u(t):

$$\frac{\partial\dot{I}_1}{\partial u} = 1, \frac{\partial\dot{I}_2}{\partial u} = -1, \frac{\partial\left(\dot{I}_1 + I_2\right)}{\partial u} = 0, \text{and } \frac{\partial\dot{u}}{\partial u} = 0 \tag{45}$$

Partial derivatives with respect to $\dot{u}$:

$$\frac{\partial\dot{I}_1}{\partial\dot{u}} = -20, \frac{\partial\dot{I}_2}{\partial\dot{u}} = 0, \frac{\partial\left(\dot{I}_1 + I_2\right)}{\partial\dot{u}} = -20, \text{and } \frac{\partial\dot{u}}{\partial\dot{u}} = 1 \tag{46}$$

4 Discussion

In Mexico, strategic planning for cocoa cultivation emphasizes the need to increase domestic production and reduce import dependency through competitive and sustainable practices. This article emphasizes the importance of technological innovation, strategic planning, and sustainability for developing sustainable agri-food systems and improving the livelihoods of cocoa producers. Despite their apparent simplicity, low-volume AFSCs present significant management challenges. Factors such as dynamic pricing, sustainability balances, and ethnographic measures for operational productivity turn these systems into complex tasks that can trigger broader geopolitical impacts over time.

Value creation in the CSC is a multi-layered process involving production improvements, efficient processing, strategic logistics, market differentiation, and sustainability practices. By enhancing quality, reducing inefficiencies, strengthening governance, and promoting social and environmental wellbeing, the CSC becomes more competitive, resilient, and profitable, especially benefiting small producers. In future research, advanced mathematical models will be developed to optimize dynamic pricing mechanisms within the cocoa supply chain using ordinary differential equations (ODEs) and partial differential equations (PDEs).

In addition, the analysis of cocoa supply chains, particularly in the context of low-volume agri-food supply chains (AFSCs), underscores the intricate balance between price dynamics, sustainability, and human resource management. While these challenges may appear straightforward on the surface, they are deeply intertwined with factors such as technological innovation, strategic planning, and competitive practices. Addressing these complexities through sustainable and resilient supply chain models is essential for reducing dependency on imports and enhancing domestic production capabilities.

Moreover, by focusing on sustainability and worker productivity, cocoa supply chains can be transformed to not only boost profitability but also improve the livelihoods of cocoa producers. Technological advancements and ethnographic measures that enhance operational efficiency are key drivers of these improvements. This technical note highlights that value creation in cocoa supply chains relies on managing the complex interactions between price dynamics, human resources, and sustainability. Even low-volume AFSCs require careful handling due to market volatility, environmental constraints, and productivity challenges. Strengthening domestic production through sustainable, competitive, and technology-driven practices is essential. Recognizing supply chains as dynamic systems enables the development of agile strategies that improve efficiency, strengthen producer livelihoods, and build resilient, future-ready agri-food systems. An AFSC covers all stages from farm to fork, and its sustainability depends on managing these stages to minimize environmental impact, support social wellbeing, and ensure economic viability. These indicators collectively reveal the structural weakness of the sector and its limited capacity to generate significant economic value, despite its historical and cultural importance.

Furthermore, sustainability refers to each state of a system in which environmental, economic, and social aspects influence the behavior of the system. In the context of cocoa DSCs, analyzing scenarios such as dynamic pricing, ethnographic factors, and sustainability impacts is essential for understanding the creation of value—particularly in low-volume systems that produce high-value final goods at the retail stage.

Finally, the stability analysis of cocoa DSCs as dynamical systems relies on distinct mathematical approaches with proper system architecture. The Routh–Hurwitz stability criterion provides a necessary and sufficient condition for the stability of linear, continuous-time, SISO systems. The sensitivity analysis demonstrate the variations between parameters for inventory levels along the cocoa DSC, with time delays, and input relations.

Data availability statement

The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation.

Author contributions

YD: Conceptualization, Formal analysis, Investigation, Methodology, Validation, Writing – original draft, Writing – review & editing. JS-L: Conceptualization, Funding acquisition, Methodology, Project administration, Resources, Writing – review & editing. ES: Conceptualization, Funding acquisition, Methodology, Resources, Supervision, Validation, Writing – review & editing. NS: Conceptualization, Formal analysis, Methodology, Resources, Visualization, Writing – original draft, Writing – review & editing.

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